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Extension of mappings of Bing spaces into ANRs

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Abstract

An atom is a hereditarily indecomposable continuum. A Bing space is a compactum in which every subcontinuum is an atom. $E \subset K$ is a sample for K if E meets every component of K. $\sigma(K) \leqslant n$ if K has a σ-compact sample E with $\dim E \leqslant n$. Hence, if K has countably many components then $\sigma(K) = 0$.

It is proved that if K is a closed subset of a Bing space X then (i) if $\sigma(K) = 0$ then every map of K in a connected ANR extends upon X; (ii) if $\sigma(K) \leqslant n$ then every map of K in S_{n+1} extends upon X.

Thus for extension of maps in Bing spaces $\sigma(K)$ may replace $\dim K$ (since both (i) and (ii) hold for every space X, not just Bing spaces, if $\dim K \leqslant n$). © 1997 Elsevier Science B.V.

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1. Atoms, Bing spaces, samples and extension

In [3] Krasinkiewicz proved that a map defined on a subcontinuum K of a hereditarily indecomposable continuum with values in an ANR (absolute neighbourhood retract) extends upon the whole continuum. This was a surprising result since by the characterization of dimension in terms of mappings to spheres [2, Theorem VI 4] in every space of dimension $n+1$ there exists a closed set which carries an unextendable map into S_n (the n-sphere) and in common examples this set is a continuum. Note also that in [1] Bing proved the existence of hereditarily indecomposable continua of all dimensions. We shall follow [6] and use the term atom for a hereditarily indecomposable continuum; a Bing space is a compactum in which every component is an atom.

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In [3, Problem 1] it is asked whether the same result holds if the set K under consideration is not a continuum, but has instead only countably many components (the ANR must then be assumed to be connected), and, in a note added in proof in [3] the author mentions that according to the referee of that paper, the answer to Problem 1 is affirmative. However, to the best of the knowledge of this author as well as the author of [3], no such result has been published. In this note we prove results which affirm the above problem and give much more information.

Krasinkiewicz's result suggests that as far as extension of maps in ANRs is concerned a subcontinuum of an atom may be considered as playing the role of a single point (note however that a map on an atom though always extendable need not a be homotopically trivial). Our result is in the same direction. Recall that if $\dim K \leqslant n$ then every map $f : K \mapsto S$ extends, where S is any connected ANR if $n = 0$ and $S = S_{n+1}$ if $n > 0$ [2, VI].

Definition 1.1. A subset E of K is a sample for K if E meets every component of K.

Definition 1.2. $\sigma(K) \leqslant n$ $(n = 0, 1, \ldots)$ if there exists a σ-compact n-dimensional sample E of K.

Remark. Definition 1.2 applies to every topological space K, but we shall consider only compact metrizable spaces.

Theorem 1.3. *Let K be a closed subset of a Bing space X.*
 (i) *If $\sigma(K) = 0$ then every map of K in a connected ANR extends upon X.*
 (ii) *If $\sigma(K) \leqslant n$, $n \geqslant 1$, then every map of K in S_{n+1} extends upon X.*

Theorem 1.3 follows from our next definition and two theorems:

Definition 1.4. Let K be a closed subset of a compact metric space X. (X, K) is $\delta(n)$ $(n = 0, 1, 2, \ldots)$ if for every closed subset H of X with $H \cap K = \emptyset$ and every $\varepsilon > 0$ there is a closed separator T in X between K and H with $d_{n+1}(T) < \varepsilon$.

Recall that $d_k(T)$ is the k-dimensional degree of T as defined in [4, p. 105]. $d_{n+1}(T) < \varepsilon$ if and only if T admits a finite open cover of mesh $< \varepsilon$ and order $\leqslant n$. Hence, for T compact $\dim T \leqslant n$ is equivalent to $d_{n+1}(T) = 0$.

Theorem 1.5. *Let K be a closed subset of a Bing space X. If $\sigma(K) \leqslant n$ then (X, K) is $\delta(n)$.*

Theorem 1.6. *Let K be a closed subset of a compact metrizable space X with (X, K) as $\delta(n)$. Then*
 (i) *If $n = 0$, then every map of K in a connected ANR extends upon X.*
 (ii) *If $n \geqslant 1$ then every map of K in S_{n+1} extends upon X.*

Note the following facts about samples $\sigma(K)$, $\delta(n)$ and extendability (K is assumed to be compact).

1.7. Theorem 1.6 applies to every compact metric space not just atoms or Bing spaces.

1.8. If K has countably many components then it clearly has a countable sample E and hence $\sigma(K) = 0$. Thus Theorem 1.3(i) affirms Problem 1 of [3].

1.9. Assume that all the components of K have the same finite dimension n. By [8, Proposition 3.1] there exists a 0-dimensional σ-compact subset E of K such that

$$\dim(K \setminus E) \leqslant n - 1.$$

E must be a sample for K and hence $\sigma(K) = 0$.

1.10. Let K^0 denote the space of components of K and let $q : K \mapsto K^0$ be the quotient map (i.e., $q(x)$ is the component of K which contains x). $E \subset K$ is a sample if and only if $qE = K^0$. If q is an open map then its inverse $q^{-1} : K^0 \mapsto K$ is a lower-semicontinuous set valued map. As $\dim K^0 = 0$ it follows form one of Michael's selection theorems [7, Theorem 2] that q^{-1} admits a continuous selection $p : K^0 \mapsto K$. Clearly p is an embedding. Hence $E = pK^0$ is a compact 0-dimensional sample for K and it follows that $\sigma(K) = 0$.

1.11. Let K_1 denote the union of all nontrivial components of K and let $K_0 = K \setminus K_1$. K_1 is σ-compact and K_0 is a 0-dimensional G_δ in K (note that $q|_{K_0} : K_0 \mapsto K^0$ is an embedding (see 1.10)). Every sample of K is of the form $E = K_0 \cup A$ where A is a sample for K_1. If $\dim K_1 = n$, n finite, then K_1 has a σ-compact $(n-1)$-dimensional sample A. Indeed, let $K_1 = Y \cup Z$ with $\dim Y = 0$ and $\dim Z = n - 1$. Let W be a 0-dimensional G_δ which contains Y. Set $A = K_1 \setminus W$. $A \subset Z$, so $\dim A \leqslant n - 1$, A is σ-compact and $\dim(K_1 \setminus A) = 0$. As every component of K_1 is of positive dimension A meets each component.

Still we may have $\sigma(K) = \dim K = n$ even when $1 \leqslant n < \infty$. To see this, let X be an $(n+1)$-dimensional atom. By [5, Proposition 2.16] there exists an n-dimensional closed subset K of X which carries an unextendable map in S_n. By Theorem 1.3(ii) $\sigma(K) = n$. Here no other sample E of K is more efficient than K itself. Moreover, if $A \subset K_1$ is the above $(n-1)$-dimensional sample for K_1 then $\dim(A \cup K_0) = n$ and hence K_0 is not σ-compact in this example.

2. Proofs

Let $f : X \mapsto Y$ be a map of compact spaces. Set $S_f = \{x \in X\colon\ f^{-1}f(x) = \{x\}\}$. S_f is a G_δ-subset of X.

Lemma 2.1. *Let E be an n-dimensional σ-compact subset of a compact space X. Let K and H be disjoint closed subsets of X. There exists an $(n+1)$-dimensional compact space Y and a monotone map $f : X \mapsto Y$ such that $E \subset S_f$ and f separates between K and H (i.e., $f(K) \cap f(H) = \emptyset$).*

Proof. By [8, Theorem 2.2] if $E_1 \subset X$ is 0-dimensional and σ-compact and if D_1 is a dendrite with a dense set of nonseparating points, then the set of maps g in $C(X, D_1)$ with $S_g \supset E_1$ is residual (i.e., complemented by a set of first category) in $C(X, D_1)$. By [8, Proposition 3.1] let $E_1 \subset E$ be 0-dimensional and σ-compact such that $\dim(E \setminus E_1) \leqslant n-1$. Let D_1 be a dendrite with a dense set of nonseparating points. By the above there exists a map $g_1 : X \mapsto D_1$ with $g_1(K) \cap g_1(H) = \emptyset$ and $S_{g_1} \supset E_1$. As S_{g_1} is a G_δ, $E \cap (X \setminus S_{g_1}) = F_{n-1}$ is an $(n-1)$-dimensional σ-compact subset of X. Now we proceed inductively: let $E_2 \subset F_{n-1}$ be a 0-dimensional F_σ with $\dim(F_{n-1} \setminus E_2) \leqslant n-2$ and let $g_2 : X \mapsto D_2$ be such that $S_{g_2} \supset E_2$. Define $F_{n-2} = F_{n-1} \cap (X \setminus S_{g_2})$ and continue by an obvious induction to obtain $g_i : X \mapsto D_i$, $1 \leqslant i \leqslant n+1$, where all D_i are dendrites. Set $g = (g_1, g_2, \ldots, g_{n+1}) : X \mapsto D_1 \times D_2 \times \cdots \times D_{n+1} = D$. Then $\dim D \leqslant n+1$ as $\dim D_i = 1$ and

$$S_g \supset \bigcup_{i=1}^{n+1} S_{g_i} \supset \bigcup_{i=1}^{n+1} E_i \supset E.$$

Also, as g_1 separates K from H, so does g. Let $g = \ell \circ f$ be the monotone light decomposition of g with $f : X \mapsto Y$ monotone and $\ell : Y \mapsto D$ light. Then $\dim Y \leqslant \dim D + \dim \ell = n+1$ and clearly $S_f \supset S_g$ and $f(K) \cap f(H) = \emptyset$. □

Lemma 2.2. *Let $f : T \mapsto L$ be a mapping of compacta such that f is an ε-mapping (i.e., diameter $f^{-1}(y) < \varepsilon$ for all y in L) and $\dim L \leqslant n$. Then $d_{n+1}(T) < \varepsilon$.*

Proof. Let $\delta > 0$ be so small that diameter $U < \delta$ in L implies that diameter $f^{-1}(U) < \varepsilon$ in T. Let C be a finite open cover of L of order $\leqslant n$ and mesh $< \delta$. Then

$$\{f^{-1}(U) \colon U \in C\}$$

has order $\leqslant n$ and mesh $< \varepsilon$. □

Proof of Theorem 1.5. Let $H \subset X$ disjoint from K and $\varepsilon > 0$ be given. As $\sigma(K) \leqslant n$ there exists a σ-compact sample E of K with $\dim E \leqslant n$. Apply Lemma 2.1. to obtain an $(n+1)$-dimensional space Y and a monotone map $f : X \mapsto Y$ with $S_f \supset E$ and $f(K) \cap f(H) = \emptyset$. Note that as X is a Bing space and f is monotone, Y is a Bing space, too. Set

$$Z = \{y \in Y \colon \operatorname{diam} f^{-1}(y) \geqslant \varepsilon\}. \tag{1}$$

Z is a closed subset of Y. Let Z^0 denote the space of components of Z and let $q : Z \mapsto Z^0$ be the quotient map (as in 1.10).

Claim 2.3. *$q(Z \cap f(K)) \cap q(Z \cap f(H)) = \emptyset$ (in Z^0).*

Proof. Equivalently: a component A of Z does not meet both $f(K)$ and $f(H)$. Indeed, assume that A meets $f(K)$. Let B be a component of K such that $f(B) \cap A \neq \emptyset$. As Y is a Bing space we must have $f(B) \supset A$ or $f(B) \subset A$ (since else $f(B) \cup A$ would have been a decomposable continuum) But $f(B) \subset A$ is impossible. To see this recall that E

is a sample for K and B is a component of K. Hence there is a point $x \in B \cap E$, and $E \subset S_f$. So $x \in B \cap S_f$. Thus $f^{-1}f(x) = \{x\}$. If $f(B) \subset A$, then $y = f(x) \in A \subset Z$ and by (1) $\operatorname{diam} f^{-1}(y) \geqslant \varepsilon > 0$—a contradiction. It follows that $f(B) \supset A$. Hence $A \cap f(H) = \emptyset$ since else $f(B)$ would have met $f(H)$.

This proves Claim 2.3. $\square$

As $\dim Z^0 = 0$ there exist clopen sets Z_1^0 and Z_2^0 in Z^0 with

$$Z^0 = Z_1^0 \cup Z_2^0, \qquad Z_1^0 \cap Z_2^0 = \emptyset,$$
$$q\big(Z \cap f(K)\big) \subset Z_1^0, \qquad q\big(Z \cap f(H)\big) \subset Z_2^0.$$

Set

$$W_1 = q^{-1}\big(Z_1^0\big) \cup f(K), \qquad W_2 = q^{-1}\big(Z_2^0\big) \cup f(H).$$

Then $W_1 \cap W_2 = \emptyset$ and $Y \supset W_1 \cup W_2 \supset Z$.

As $\dim Y \leqslant n+1$ there exists a closed n-dimensional subset L of Y which separates W_1 from W_2. Set $T = f^{-1}(L)$. T separates K from H in X, and as $L \cap Z = \emptyset$, $f|_T : T \mapsto L$ is an ε-map. Lemma 2.2 now implies that $d_{n+1}(T) < \varepsilon$ and we are done. $\square$

Proof of Theorem 1.6. Let (X, K) be $\delta(n)$, $n = 0, 1, \ldots$, and let $f : K \mapsto S$ be a map where, if $n = 0$, S is any connected ANR and if $n \geqslant 1$, $S = S_{n+1}$. As S is an ANR there exists an open subset U of X such that $K \subset U$ and an extension $f_1 : U \mapsto S$ of f. Let $V \subset X$ be open such that $K \subset V \subset \overline{V} \subset U$. By compactness $f_1|_{\overline{V}}$ is uniformly continuous. Let $\delta = \delta(S) > 0$ be small enough such that if two maps $g, h : Y \mapsto S$ on some compactum Y satisfy $d\big(g(y), h(y)\big) < \delta$ for all $y \in Y$ then g and h are homotopic (note that for S_n with the usual Euclidean metric $\delta(S_n) = 2$). Let $\varepsilon > 0$ be such that the variation of f_1 on subsets of $\overline{V}$ with diameter $< \varepsilon$ is less than $\delta/2$. Let T be a closed set with $d_{n+1}(T) < \varepsilon$ which separates between K and $X \setminus V$. Thus $X \setminus T = A \cup B$, $K \subset A \subset V$, $X \setminus V \subset B$. Set $f_2 = f_1|_T$. f_2 is homotopically trivial. Indeed, if $n = 0$, $d_1(T) < \varepsilon$ implies that T can be covered by finitely many mutually disjoint clopen sets of diameter $< \varepsilon$. Let $g : T \mapsto S$ be defined to be constant on each of these clopen sets U, with $g(U) = f_2(x)$ for some arbitrary $x \in U$. Then $d\big(f_2(t), g(t)\big) < \delta$ for all t in T which implies that f_2 is homotopic to g, while g is null-homotopic since it has a finite range in S.

For $n \geqslant 1$ let $\{U_i\}_{i=1}^m$ be an open cover of T with mash $< \varepsilon$ and order n. Let $y_i \in f_2(U_i) \subset S_{n+1}$ and let $\{h_i\}_{i=1}^m$ be a partition of unity subordinated to the cover $\{U_i\}_{i=1}^m$ (i.e., $h_i(T \setminus U_i) = 0$). Let

$$g(t) = r \sum_{i=1}^{m} h_i(t) y_i$$

where $r : R^{n+2} \setminus \{0\} \mapsto S_{n+1}$ is the radial projection. Then $\|f_2(t) - g(t)\| < \delta$ for all t in T (norm in R^{n+2}) so $f_2 \cong g$. Also, the range $g(T)$ of g in S_{n+1} is at most n-dimensional (since the order of $\{U_i\}_{i=1}^m$ is n and hence at most $n+1$ of $h_i(t)$ are positive for each $t \in T$). Hence $g(T)$ does not cover S_{n+1} which implies that $g \cong 0$.

From Borsuk's Homotopy Extension Theorem [2, Theorem VI 5] it follows that f_2 admits an extension $f_3 : B \cup T \mapsto S_{n+1}$. Then

$$f_4 = \begin{cases} f_2 & \text{on } A \cup T, \\ f_3 & \text{on } B \cup T \end{cases}$$

is a continuous extension of f upon X. $\square$

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